# q-trace and gauge invariants of the quantum group $GL_q(2)$ gauge fields



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# $q$-trace and gauge invariants of the quantum group $GL_q(2)$ gauge fields

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**Abstract.** In this paper, it is proved that if a suitable requirement of commutation relations between the entries of the quantum transformation matrix and of the gauge potential matrix is satisfied, then the $q$-trace of quantum group $GL_q(2)$ gauge field intensity is gauge invariant, which is a $q$-analytic function on the quantum plane.

In recent years, there have been several approaches to quantum group gauge field theories [1–3], however, there are still some difficulties. It seems to us the crux of the problem is that the algebraic generators of the quantum hyperplanes do not play the roles of non-commutative movable coordinates as in ordinary classical analysis as yet. For this reason, in the preceding paper [4] we suggested a fundamental way to overcome the difficulties, i.e. a non-commutative analysis on a quantum hyperplane is given, and it has been used in the discussions concerning quantum group gauge fields. However, an important problem—the gauge invariant problem—remains to be settled. In this paper some results concerning this problem are given; at the present only the two-dimensional case is considered.

It is known that in seeking the invariants, a useful formula is

$$\mathrm{Tr}(SMS^{-1}) = \mathrm{Tr}(M) \tag{1}$$

where Tr denotes the trace of a matrix, and the entries of matrices $S$ and $M$ are numbers, or at the least they are commutative with each other in multiplication. For the case of quantum groups (quantum matrices), equation (1) does not hold in general. However, recently, Iseav and Popowicz [3] obtained a result for a $2\times 2$ matrix $T=(T^i_j)\in GL_q(2)$ and a matrix $F=(F^k_l)$ if $T^i_j$s commute with $F^k_l$s:

$$[T^i_j, F^k_l] = 0 \qquad (i,j,k,l=1,2) \tag{2}$$

then

$$\mathrm{Tr}_q(F) \equiv q^{-1}F^1_1 + qF^2_2 = \mathrm{Tr}_q(TFT^{-1}) \tag{3}$$

where $q\in\mathbb{C}$ is the deformation parameter, and we suppose it is not a unit root in this paper. $\mathrm{Tr}_q$ is called the $q$-trace. In this paper, we consider how to use (3) in the non-commutative analysis on the quantum plane [5] $\mathscr{C}=\mathbb{C}\langle x,y\rangle$ with the commutation relation

$$xy = qyx. \tag{4}$$

There has been some criticism [2] of the result in [3], however the result in this paper is not effected.

In the following the Greek indices $\alpha, \beta, \gamma, \ldots$ take values $0, 1, 2, \ldots,$ and the Latin indices $i, j, k, \ldots$ take values 1, 2. We use the Einstein summation convention, i.e. the repeated Greek/Latin indices are summed over the values $0, 1, 2, \ldots/1, 2$, unless there is a contrary state. In addition, we call the finite or infinite generator set of a non-commutative and associative algebra a $q$-sequence. Now, we consider a non-commutative algebra $\mathscr{F}$, which is freely generated by the $q$-sequence $\{f_{\alpha\beta}, g_{\gamma\delta}\}$ with the commutation relations

$$\begin{aligned} &q^{-\beta\gamma} f_{\alpha\beta} g_{\gamma\delta} = q^{1-\alpha\delta} g_{\gamma\delta} f_{\alpha\beta} \\ &[f_{\alpha\beta}, x] = [f_{\alpha\beta}, y] = [g_{\gamma\delta}, x] = [g_{\gamma\delta}, y] = 0 \end{aligned} \tag{5}$$

(no summing for the repeated indices).

We consider the infinite polynomials $f^q(x, y) = f_{\alpha\beta} x^\alpha y^\beta$ and $g^q(x, y) = g_{\gamma\delta} x^\gamma y^\delta$ which can be regarded as elements of the algebra $\mathscr{F} \otimes \mathscr{C}$. It is easily verified using (4) and (5), that $f^q$, $g^q$ satisfy the commutation relation

$$f^q g^q = q g^q f^q. \tag{6}$$

This means that by $f^q$ and $g^q$ a quantum plane is constructed again. $f^q$ and $g^q$ can be called $q$-analytic functions [4], since they change into ordinary analytic functions when $q \to 1$ and $x, y$ are taken as ordinary complex variables, and $f_{\alpha\beta}$ and $g_{\gamma\delta}$ are taken as complex numbers.

For a $q$-analytic function $f^q$, we define the partial derivatives by

$$\begin{aligned} &\partial_x f^q = q^{2\beta} [\alpha]_q f_{\alpha\beta} x^{\alpha-1} y^\beta \\ &\partial_y f^q = q^\alpha [\beta]_q f_{\alpha\beta} x^\alpha y^{\beta-1} \end{aligned} \tag{7}$$

where the $q$-integer $[\alpha]_q = (q^{2\alpha} - 1)/(q^2 - 1) = q^{2(\alpha-1)} + q^{2(\alpha-2)} + \ldots + 1$. Let

$$\mathrm{d} = \mathrm{d}x\, \partial_x + \mathrm{d}y\, \partial_y \tag{8}$$

then differential calculus [6] on the quantum $\mathscr{C}$ gives

$$\begin{aligned} &\mathrm{d}^2 = 0 \\ &\mathrm{d}(f^q g^q) = (\mathrm{d}f^q) g^q + f^q\, \mathrm{d}g^q \\ &\mathrm{d}(\Phi\Omega) = \mathrm{d}\Phi\Omega + (-1)^k \Phi\, \mathrm{d}\Omega \end{aligned} \tag{9}$$

where $\Phi$ is a $k$-form. In this paper we use the following commutation relations [6]

$$\begin{aligned} &x\,\mathrm{d}x = q^2\,\mathrm{d}x\, x \qquad x\,\mathrm{d}y = (q^2 - 1)\,\mathrm{d}x\, y + q\,\mathrm{d}y\, x \\ &y\,\mathrm{d}x = q\,\mathrm{d}x\, y \qquad y\,\mathrm{d}y = q^2\,\mathrm{d}y\, y \\ &\mathrm{d}x\,\mathrm{d}x = \mathrm{d}y \cdot \mathrm{d}y = 0 \qquad \mathrm{d}x\,\mathrm{d}y = -\frac{1}{q}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{10}$$

According to [4], we can consider the $2 \times 2$ matrix

$$T(x, y) = [T_b^m(x, y)] \equiv \begin{bmatrix} A(x, y) & B(x, y) \\ C(x, y) & D(x, y) \end{bmatrix} \qquad (m, b = 1, 2) \tag{11}$$

where $T_b^m(x, y) = (T_b^m)_{\alpha\beta} x^\alpha y^\beta$ is a $q$-analytic function on $\mathscr{C}$. If some commutation relations similar to (5) in the $q$-sequence $\{A_{\alpha\beta}, B_{\alpha\beta}, C_{\alpha\beta}, D_{\alpha\beta}\}$ are satisfied (see [4]), then we have

$$\begin{aligned}&AB=qBA \qquad AC=qCA\\&AD-DA=(q-q^{-1})BC\\&BC=CB \qquad BD=qDB\\&CD=qDC.\end{aligned} \tag{12}$$

This means that $T(x, y)$ is a $q$-analytic matrix on the quantum plane, and is an element of the quantum group $GL_q(2)$. In [4] $T$ is taken as the gauge transformation, and the gauge potential is taken as a $2\times 2$ 1-form matrix $E(x, y)=dx^i E_i$, ($dx^1=dx$, $dx^2=dy$)

$$E_i=E_i(x, y)=\begin{bmatrix}E_{i1}^1 & E_{i2}^1\\ E_{i1}^2 & E_{i2}^2\end{bmatrix} \qquad (i=1, 2) \tag{13}$$

where the entries $E_{ia}^k(x, y)=(E_{ia}^k)_{\alpha\beta}x^\alpha y^\beta$, ($k$, $a=1, 2$) are $q$-analytic functions. The key problem is how to determine the interior algebraic structure of $q$-sequence $\{(E_{ia}^k)_{\alpha\beta}\}$ and the algebraic relation between $q$-sequences $\{(E_{ia}^k)_{\alpha\beta}\}$ and $\{(T_b^m)_{\gamma\delta}\}$. The interior algebraic structure will be determined by the gauge field equations as in [4]. As for the algebraic relation between $\{(E_{ia}^k)_{\alpha\beta}\}$ and $\{(T_b^m)_{\gamma\delta}\}$, we notice that it is not yet given.

Now, we define the commutation relation between $\{(E_{ia}^k)_{\alpha\beta}\}$ and $\{(T_b^m)_{\gamma\delta}\}$ by the following equation

$$dx^i F_{ija}^k(x, y)\, dx^j T_b^m(x, y)=T_b^m(x, y)\, dx^j F_{ija}^k(x, y)\, dx^j \tag{14}$$

where $F_{ij}=(F_{ija}^k)$ ($i, j=1, 2$) is the gauge field intensity [4], which is a $2\times 2$ matrix, and is anti-symmetric for the indices $i, j$:

$$F_{ij}=\partial_i Q_j^k(E_k)-\partial_j Q_i^k(E_k)+E_i Q_j^k(E_k)-E_j Q_i^k(E_k) \qquad (\partial_1=\partial_x, \partial_2=\partial_y). \tag{15}$$

Here and in the following, $O_i^k$ and $Q_i^k$, respectively, denote the operators left and right translating $dx^k$ through $q$-analytic functions, which are linear [4, 6]:

$$\begin{aligned}&f^q\, dx^k=dx^i O_i^k(f^q)\\&dx^k f^q=Q_i^k(f^q)\, dx^i.\end{aligned} \tag{16}$$

Evidently, we have

$$O_i^s\circ Q_s^k=Q_i^s\circ O_s^j=\delta_i^k\circ 1 \tag{17}$$

where 1 denotes the identical operator. From (10) we can concretely calculate the results as follows:

$$\begin{aligned}&O_1^1(x^\alpha y^\beta)=q^{2\alpha+\beta}x^\alpha y^\beta \qquad O_2^1(x^\alpha y^\beta)=0\\&O_1^2(x^\alpha y^\beta)=q^{2\beta}(q^{2\alpha}-1)x^{\alpha-1}y^{\beta+1} \qquad O_2^2(x^\alpha y^\beta)=q^{\alpha+2\beta}x^\alpha y^\beta\\&Q_1^1(x^\alpha y^\beta)=q^{-2\alpha-\beta}x^\alpha y^\beta \qquad Q_2^1(x^\alpha y^\beta)=0\\&Q_1^2(x^\alpha y^\beta)=q^{1-3\alpha-\beta}(1-q^{2\alpha})x^{\alpha-1}y^{\beta+1} \qquad Q_2^2(x^\alpha y^\beta)=q^{-\alpha-2\beta}x^\alpha y^\beta.\end{aligned} \tag{18}$$

Therefore (14) can be written as

$$Q_r^i[F_{ija}^k \cdot Q_s^j(T_b^m)]\,\mathrm{d}x^r\,\mathrm{d}x^s = T_b^m \cdot Q_p^i(F_{ita}^k)\,\mathrm{d}x^p\,\mathrm{d}x^t. \tag{19}$$

Next, by using (10), (18) and the anti-symmetry of $F_{ij}$ concerning the indices $i$ and $j$, and from the coefficient of the term $\mathrm{d}x\,\mathrm{d}y$ we obtain

$$Q_1^1[F_{12a}^k Q_2^2(T_b^m)] + \frac{1}{q} Q_2^2[F_{12a}^k Q_1^1(T_b^m)] = T_b^m\left[Q_1^1(F_{12a}^k) + \frac{1}{q} Q_2^2(F_{12a}^k)\right]. \tag{20}$$

In order to obtain the explicit formulation of the commutation relation between $q$-sequences $\{(F_{ija}^k)_{\gamma\delta}\}$ and $\{(T_b^m)_{\alpha\beta}\}$, we must use (7), (15) and (18), and after lengthy calculation we see that (20), in fact, can be induced into the following commutation relation:

$$q^{-\beta\gamma}(T_b^m)_{\alpha\beta}(F_{12a}^k)_{\gamma\delta} = q^{-3(\alpha+\beta)-\alpha\delta}(F_{12a}^k)_{\gamma\delta}(T_b^m)_{\alpha\beta} \tag{21}$$

(no summing for the repeated indices) where $F_{12}$ is the only essential non-zero component of $F_{ij}$:

$$\begin{aligned}
F_{12} &= (F_{12})_{\gamma\delta}x^\gamma y^\delta \\
(F_{12})_{\gamma\delta} &= q^{-2\gamma-\delta-1}[\delta+1]_q(E_1)_{\gamma,\delta+1} \\
&\quad - \{q^{-\gamma-1}[\gamma+1]_q + q^{-2-3\gamma-\delta}(1-q^{2\gamma+2})[\delta+1]_q\}(E_2)_{\gamma+1,\delta} \\
&\quad + \sum_{\substack{\alpha+\rho=\gamma\\ \beta+\tau=\delta}} \{q^{-\alpha-2\beta-\beta\rho}(E_1)_{\alpha\beta}(E_2)_{\rho\tau} - q^{-2\rho-\tau-\beta\rho}(E_2)_{\alpha\beta}(E_1)_{\rho\tau}\} \\
&\quad + \sum_{\substack{\alpha+\rho=\gamma+1\\ \beta+\tau=\delta-1}} q^{1+\beta-3\rho-\tau-\beta\rho}(1-q^2)(E_2)_{\alpha\beta}(E_2)_{\rho\tau}
\end{aligned} \tag{22}$$

and $(E_i)_{\alpha\beta}$ is the matrix whose entries are $(E_{ia}^k)_{\alpha\beta}$ $(k, a=1,2)$. Therefore if we stipulate that the $q$-sequences $\{(F_{ija}^k)_{\gamma\delta}\}$ and $\{(T_b^m)_{\alpha\beta}\}$ obey the commutation relation (21), then we have

$$[\mathrm{d}x^i\,F_{ija}^k\,\mathrm{d}x^j, T_b^m] = 0. \tag{23}$$

According to [4], the gauge transformation of $E$ is

$$E \to \tilde{E} = TET^{-1} - (\mathrm{d}T)T^{-1} \tag{24}$$

where $T^{-1} \in \mathrm{GL_q}(2)$ is the inverse of $T$,

$$T^{-1} = \frac{1}{\det_q(T)}\begin{pmatrix} D & -\frac{1}{q}B \\ -qC & A \end{pmatrix} \qquad \det_q(T) = AD - q\,BC \tag{25}$$

and the gauge transformation of $F_{ij}$ has been proved, i.e.

$$F_{ij} \to \tilde{F}_{ij} = Q_i^r(T)F_{rs}Q_j^s(T^{-1}). \tag{26}$$

Now, left and right multiply the two sides of (26) by $\mathrm{d}x^i$ and $\mathrm{d}x^j$, respectively, and take the sum, then we have

$$\begin{aligned} &M \to \tilde{M} = TMT^{-1} \\ &M = \mathrm{d}x^i\,F_{ij}\,\mathrm{d}x^j \qquad \tilde{M} = \mathrm{d}x^i\,\tilde{F}_{ij}\,\mathrm{d}x^j. \end{aligned} \tag{27}$$

Equation (23) is just

$$[M_a^k, T_b^m]=0. \tag{28}$$

Therefore according to (2) and (3), we have

$$\mathrm{Tr}_q(M)=\mathrm{Tr}_q(\tilde{M}). \tag{29}$$

From the anti-symmetry of $F_{ij}$ and (10) and (16) we have

$$\left(O_2^2+\frac{1}{q}O_1^1\right)\mathrm{Tr}_q(F_{12}-\tilde{F}_{12})=0 \tag{30}$$

i.e.

$$q^{\alpha+\beta}(q^{\beta}+q^{\alpha-1})\,\mathrm{Tr}_q[(F_{12})_{\alpha\beta}-(\tilde{F}_{12})_{\alpha\beta}]=0. \tag{31}$$

This means that

$$\mathrm{Tr}_q(F_{ij})=\mathrm{Tr}_q(\tilde{F}_{ij}) \tag{32}$$

therefore on the quantum plane $\mathscr{C}$ the $q$-analytic function

$$I_{ij}(x,y)=q^{-1}F_{ij1}^1(x,y)+qF_{ij2}^2(x,y) \tag{33}$$

is a $GL_q(2)$ gauge invariant. This result is different from [3], since $I_{ij}$ is not a 'constant' element of $GL_q(2)$.

When $q\to 1$, then $O_i^k$ and $Q_i^k\to\delta_i^k\cdot 1$, and the above results all return to the corresponding results in the ordinary gauge field theories, i.e. we obtain a quantum analogy of the ordinary gauge invariants.

Summing up, we have proved that when some suitable commutation relations are provided, then there are gauge invariants in the quantum group $GL_q(2)$ gauge fields. As for the case of higher dimensional quantum group $GL_q(n)$ ($n\geqslant 3$), this is more complex. The results concerned will be discussed elsewhere.

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